Sains Malaysiana 43(8)(2014): 1259–1262

A Third Order Nakashima Type Implicit Pseudo Runge-Kutta Method for Delay Differential Equations

(Kaedah Runge-Kutta Nakashima Jenis 'Pseudo' Bertahap Tiga untuk Persamaan Perbezaan Lengah)

LIM TIAN HWEE*

ABSTRACT

A third order Nakashima type implicit Pseudo Runge-Kutta method is presented. The free parameter was determined by minimizing the error bound. The stability region of the method was presented. Some problems on delay differential equations are tested to compare the accuracy of the proposed method with third order RADAU I.

Keywords: Delay differential equations; implicit pseudo Runge-Kutta method; third order

ABSTRAK

Satu kaedah Runge-Kutta Nakashima tersirat jenis "pseudo" bertahap tiga telah diterbitkan. Parameter bebas telah ditentukan dengan meminimumkan batas ralat. Rantau kestabilan kaedah tersebut juga dipersembahkan. Beberapa soalan persamaan pembezaan lengah telah diuji untuk dibanding kejituan kaedah yang diteerbitkan dengan RADAU I bertahab tiga.

Kata kunci: Bertahap tiga; kaedah Runge-Kutta jenis 'pseudo' tersirat; persamaan perbezaan lengah

INTRODUCTION

The first order delay differential equation (DDE) with one delay term can be written as:

$$\left.\begin{array}{ll} y'(t) = f(t, y(t), y(t-\tau)) & t \geq t_0 \\ y(t) = \varphi(t) & t \leq t_0 \end{array}\right\}, \tag{1}$$

where $\varphi(t)$ is the initial function. τ is, in term of t and $y(t)$, called the delay argument and $y(t-\tau)$ is the solution of the delay term.

In recent years, research has been carried out to solve DDEs using explicit or implicit Runge-Kutta (RK) method with Hermite interpolation. Such work can be found in Bellen and Zennaro (2003), Ismail et al (2002), Karoui (1992) and Orbele and Pesch (1981). However, Yaacob et al. (2011) studied the numerical treatment of DDEs using Pseudo Runge-Kutta method (PRK).

In this paper, we discussed the derivation of a third order implicit PRK method in the next section. The proposed method is compared with other third order implicit method.

IMPLICIT PSEUDO RUNGE-KUTTA METHOD

Nakashima (1982) introduce a new type of PRK method. The PRK can be written as:

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=0}^{s} b_i K_i \\ K_i &= f\left(t_n + c_i h, y_i + \lambda_i\left(y_n - y_{n-1}\right) + h\sum_{j=0}^{i} a_{ij} K_j\right) \\ c_i &= \lambda_i + \sum_{j=0}^{i-1} a_{ij} \qquad i = 2, \ldots, s, \end{aligned} \tag{2}$$

for $c_0 = \lambda_0 = -1$, $c_1 = \lambda_1 = 0$ and $0 \leq c_s \leq 1$.

From Nakashima (1982) and Shintani (1981), the following eight order conditions, which are required to construct a third order PRK method.

TABLE 1. Third order Pseudo-Runge-Kutta order conditions

$\tau_1^{(1)} : \sum_i b_i = 1$
$\tau_1^{(2)} : \sum_i b_i c_i = \frac{1}{2}$
$\tau_1^{(3)} : \sum_i b_i c_i^2 = \frac{1}{3}$
$\tau_2^{(3)} : \sum_i b_i \lambda_i + 2\sum_{ij} b_i a_{ij} c_j = \frac{1}{3}$

From Table 1, there are only four equations to be satisfied and we have 6 unknowns, thus we have two arbitrary parameters to determine. After solving all the related equations, we have:

$$\begin{aligned} b_0 &= -\frac{1}{6}\frac{3c_2 - 2}{c_2 + 1}, \qquad b_1 = \frac{1}{6}\frac{9c_2 - 5}{c_2} \\ b_2 &= \frac{5}{6}\frac{1}{c_2\left(c_2 + 1\right)}, \qquad \lambda_2 = -c_2^2 - 2a_{20} + 2a_{22}c_2, \end{aligned} \tag{3}$$

where c_2, a_{20} and a_{22} are free parameters which we want to determine.

TABLE 2. Error factors for third order pseudo Runge-Kutta method

$\tau_1^{(4)} : \sum_i b_i c_i^3 = \frac{1}{4}$
$\tau_2^{(4)} : \sum_i b_i c_i \lambda_i + 2\sum_{ij} b_i c_i a_{ij} c_j = \frac{1}{4}$
$\tau_3^{(4)} : \sum_i b_i \lambda_i + 3\sum_{ij} b_i a_{ij} c_j^2 = \frac{1}{4}$
$\tau_4^{(4)} : -\sum_{ij} b_i c_j^2 a_{ij} - \sum_i b_i \lambda_j + 6\sum_{ij} b_i a_{ij} a_{jk} c_k = 0$

All four error factors will become zero for $c_2 = 0.7$, $a_{20} = 0.833$ and $a_{22} = 0$. However, this also means for this choice of c_2, a_{20} and a_{22}, our third order method would actually become a fourth order method. However, by choosing c_2 for a value close to 0.7, such as $c_2 = \frac{33}{47}$, we expect that the error factor will become small.

We use the notation from Lotkin (1951) and Ralston (1962) to determine the error bounds E for our third order PRK method.

$$|E| \le CML^3h^4, \tag{4}$$

where C is the error constant in a region $\mathbb{R}$ about (t_n, y_n)

$$|f(x,y)| < M \text{ and} \left|\frac{f^{i+j}(x,y)}{\partial x^i \partial y^j}\right| < \frac{L^{i+j}}{M^{j-1}}, \tag{5}$$

where L and M are positive constants independent of t, y. With $c_2 = \frac{33}{47}$, the constant C is estimated by

$$\begin{aligned} C = &\left|\frac{5}{846}\right| + \left|-\frac{7}{18} + \frac{2209}{4752}a_{20} + \frac{193}{144}a_{22}\right| \\ &+ \left|-\frac{7}{36} + \frac{2209}{9504}a_{20} + \frac{193}{288}a_{22}\right|. \end{aligned} \tag{6}$$

Our objective was to minimize the right hand side of (6). We found that the bound of C is minimized when a_{20}= and $a_{22} = \frac{56}{193}$. Substituting c_2, a_{20} and a_{22} into (3), we obtain a two-stages third order pseudo Runge-Kutta method:

$$y(t_i + h) = y_i + h\left(-\tfrac{1}{96}k_0 + \tfrac{31}{99}k_1 + \tfrac{2209}{3168}k_2\right),$$

where

$$\begin{aligned} k_0 &= f(t_{i-1}, y_{i-1}) \\ k_1 &= f(t_i, y_i) \\ k_2 &= f\left(t_{i-1} + \frac{33}{47}h, y_i - \frac{36465}{426337}(y_i - y_{i-1}) \right. \\ &\quad \left. + \frac{212104}{426337}hk_1 + \frac{56}{193}hk_2\right). \end{aligned} \tag{7}$$

We present our new implicit PRK method in the tableau.

TABLE 3. Third order Implicit PRK method

-1	-1			
0	0			
$\frac{33}{47}$	$-\frac{36465}{426337}$	0	$\frac{212104}{426337}$	$\frac{56}{193}$
		$-\frac{1}{96}$	$\frac{31}{99}$	$\frac{2209}{3168}$

The local truncation error for formula (7) satisfies:

$$|E| \le \tfrac{5}{846}ML^3h^4. \tag{8}$$

STABILITY ANALYSIS

To determine the stability function of the proposed method, we applied the famous Dahlquist's test equation:

$$y' = f(x, y) = \lambda y, \tag{9}$$

to formula (7). The stability polynomial for the proposed pseudo-Runge-Kutta method is:

$$y_{i+i} = -\tfrac{1}{12}\left(\frac{2316y_i + 1530h\lambda\, y_i + 114h\lambda\, y_{i-1} + 7h^2\lambda^2 y_{i-1} + 593h^2\lambda^2 y_i}{-193 + 56h\lambda}\right). \tag{10}$$

On assuming $h\lambda = z$, $y_{i+1} = \zeta$, $y_i = \zeta^0$ and $y_{i+1} = \zeta^{-1}$, (10) becomes:

$$\zeta + \frac{2316 + 1530 + \frac{114z}{\zeta} + \frac{7z^2}{\zeta} + 593z^2}{12(-193 + 56h\lambda)} = 0. \tag{11}$$

Solving (11) we have two roots

$$\begin{aligned} \zeta_1 &= \frac{-2316 - 1530z - 593z^2 + \sqrt{\sigma}}{2(-2316 + 672z)} \\ \zeta_2 &= \frac{-2316 - 1530z - 593z^2 - \sqrt{\sigma}}{2(-2316 + 672z)}, \end{aligned} \tag{12}$$

where

$$\sigma = 5363856 + 8143056z + 4846092z^2 + 1795764z^3 + 351649z^4.$$

Since $|\zeta_1| \le 1$ and $|\zeta_2|$ are two stability functions for the new PRK method. By taking $z = x + yi$, we plot the stability region using MAPLE package. The shaded region is the region which satisfies the condition $|\zeta_1| \le 1$ and $|\zeta_2| \le 1$. The stability region for the new method is given in Figure 1.

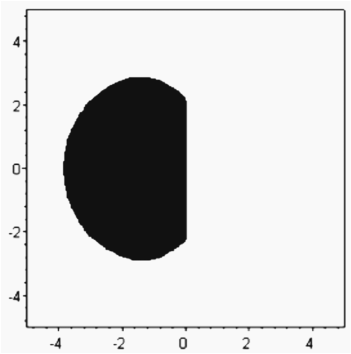

FIGURE 1. Stability region the third order PRK method (2.8)

NUMERICAL METHOD FOR SOLVING DDES

Since the method we proposed in earlier section with a third order two stages method, we compare our methods with a two stages implicit RK method show in Table 3.

TABLE 4. RADAU I

0		
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{1}{4}$	$\frac{3}{4}$

According to Ismail et al. (2002), most numerical methods for solving ordinary differential equation (ODE) can be adapted to solve DDE. Thus, DIRK method in Table 3 can be generalized into the following form when solving DDE (1),

$$y_{n+1} = y(t_n + h) = y_n + h\sum_{i=1}^{5} b_i K_i$$

$$K_i = f\left(t_i + c_i h, y_n + \sum_{j=1}^{i} a_{ij} K_j, y(t_n + c_i h - \tau)\right). \tag{13}$$

To approximate the delay term, $y(t_n + c_i h - \tau)$ on $[t_i, t_j]$, we use 3-point Hermite interpolation.

NUMERICAL RESULTS

We have written experimental program in MAPLE to solve some delay differential equations using implicit Runge-Kutta method and the method proposed in the previous section. Below are some of the test problems.

Problem 1: (Ismail et al. 2002)

$y'(t) = -y(t - \tau) + \sin(t - 1 + e^{-t}) \cos(t), t > 0$
$y(t) = \sin(t) \quad t \leq 0$
$\tau(t,y) = 1 - e^{-t}$
Exact solution: $y(t) = \sin(t), t > 0$
Results are given for $t \in [0,5]$

Problem 2: (Ismail et al. 2002)

$y'(t) = \cos(t)\, y(t-\tau), \quad t > 0$
$y(t) = 1 \quad t \leq 0$
$\tau(t,y) = t - y(t) + 2$
Exact solution: $y(t) = \sin(t) + 1$
Results are given for $t \in [0,10]$

Problem 3: (Bellen & Zennaro 2003)

$y'(t) = \dfrac{1}{2\sqrt{t}} y(t - \tau), \quad t > 1$
$y(t) = 1, \quad t \leq 1$
$\tau(t,y) = y(t) - \sqrt{2} + 1$
Exact solution: $y(t) = \sqrt{t}, \quad 1 \leq t \leq 2$
Results are given for $t \in [1,2]$

We solved the above delay differential equations using PRK proposed in the previous section and an implicit Runge-Kutta method describe in Table 4 with tolerance, TOL=0.01 and 5 Newton iterations to get the stages values. The delay term is evaluated using 3 point Hermite interpolation. The notations used are as follows:

H : Stepsize
M1 : Using Implicit PRK in Table 3
M2 : Using RADAU I in Table 4
MaxErr : Maximum error $|y(x_i) - y_i|$

The notation 6.2823577 (-7) means 6.2823577×10^{-7}.

TABLE 5. Maximum absolute error for Problem 1

H	Method	MaxErr
0.1	M1	4.5696171 (-7)
	M2	7.7241505(-6)
0.01	M1	4.9170357 (-10)
	M2	7.7795651 (-9)

TABLE 6. Maximum absolute error for Problem 2

H	Method	MaxErr
0.1	M1	6.2935352 (-6)
	M2	9.2600280 (-6)
0.01	M1	5.9143795 (-10)
	M2	9.2592766 (-9)

TABLE 7. Maximum absolute error for Problem 3

H	Method	MaxErr
0.1	M1	1.4383307 (-6)
	M2	3.1670842 (-7)
0.01	M1	6.1223481 (-11)
	M2	1.4304856 (-9)

The notation 6e-07 means 6×10^{-6}.

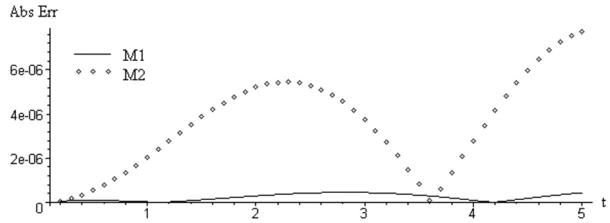

FIGURE 2. Absolute error for Problem 1 using H=0.1

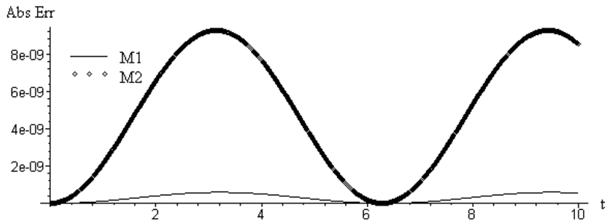

FIGURE 5. Absolute error for Problem 2 using H=0.01

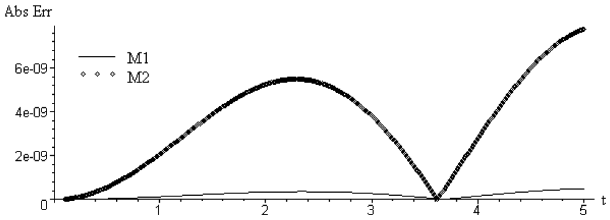

FIGURE 3. Absolute error for Problem 1 using H=0.01

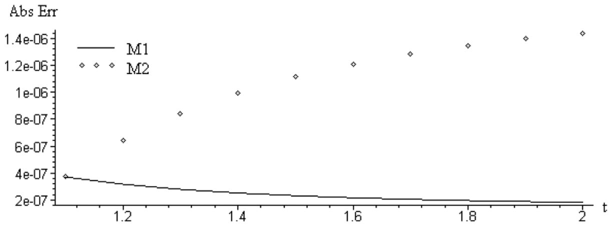

FIGURE 6. Absolute error for Problem 3 using H=0.1

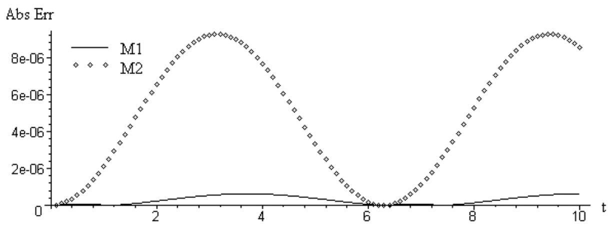

FIGURE 4. Absolute error for Problem 2 using H=0.1

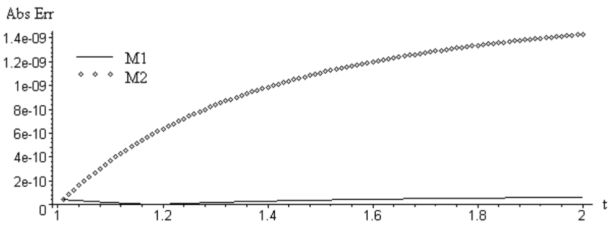

FIGURE 7. Absolute error for Problem 3 using H=0.01

CONCLUSION

In this paper, we derived an implicit third order PRK to solve DDEs. The proposed method is then compared with RADAU I. The delay term is approximated using 3-point Hermite interpolation. From all of the problems we test, the proposed method give better results than RADAU I.

REFERENCES

Bellen, A. & Zennaro, M. 2003. *Numerical Methods for Delay Differential Equations*. Oxford: Clarendon Press.

Karoui, A. 1992. On the numerical solution of delay differential equations. Master Thesis. University of Ottawa, Ontario (Unpublished).

Ismail, F., Al-Khasawneh, R.A., Aung, S.L. & Suleiman, M. 2002. Numerical treatment of delay differential equations by Runge-Kutta method using Hermite interpolation. *Matematika UTM* 18(2): 79-90.

Lotkin, M. 1951. On the accuracy of Runge-Kutta methods. *Mathematical Tables and Other Aids to Computatio*n 5(35): 128-133.

Nakashima, M. 1982. On Pseudo-Runge-Kutta Methods with 2 and 3 stages. *Publ. RIMS, Kyoto Univ.* 18: 895-909.

Orbele, H.J. & Pesch, H.J. 1981. Numerical treatment of delay differential equations by Hermite interpolation. *Numer. Math.* 31: 235-255.

Ralston, A. 1962. Runge-Kutta methods with minimum error bounds. *Math. Comp*. 16: 431-437.

Shintani, H. 1981. On pseudo-Runge-Kutta Methods of the third kind. *Hiroshima Math*. J. 11: 247-254.

Yaacob, N., Sabri, A.A., Samsudin, N. & Lim, T.H. 2011. *Application of Nakashima's 2 stages 4th order Pseudo-Runge-Kutta method and 3 stages 5th order Pseudo-Runge-Kutta Method to Delay Problems*. UTM Technical Report, LT/M BIL.6/2011.

Foundation in Business
University of Reading Malaysia
Level 7, Menara Kotaraya, Jalan Trus
80000 Johor Bahru, Johor
Malaysia

*Corresponding author; email: t.h.lim@reading.ac.uk

Received: 24 July 2012
Accepted: 6 November 2013